die Verallgemeinerung der experimentellen Ergebnisse auf diesem Gebiet.

Die Anwendung derselben Methode beim Studium der Wärmeaustauscher durch Prof. Bošnjaković hat dieselben Vorteile und erklärt die vielseitigen Möglichkeiten die sich beim Gebrauch der Betriebscharakteristik ϕ ergeben. Diese Vorteile gehen aus der angeführten Arbeit [7], welche der Berechnung von Wärmeaustauschern neue Möglichkeiten eröffnet, klar hervor und beweisen die Überlegenheit der Berechnungsmethode auf Grund der kriteriellen Gleichung, gegenüber der Methode die sich auf die mittlere Temperaturdifferenz stützt, welche auch heute, trotz der ihr gesetzten Grenzen noch sehr verbreitet ist.

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RADIATIVE TRANSFER THROUGH A SCATTERING, ABSORBING LAYER*

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NOMENCLATURE

- a, absorption coefficient;
- I, forward radiative flux;
- I_i, incident flux at the front surface;
- J, backward radiative flux;
- L, thickness of layer;
- n, index of refraction;
- s, scattering coefficient;
- x, distance from front surface;
- α, absorptance of layer;
- β, optical constant equal to $[a/(a + 2s)]^{\frac{1}{2}}$;
- γ, extinction coefficient equal to $(a + s)$;
- ρ, reflectance of layer;
- σ, optical constant equal to $[a(a + 2s)]^{\frac{1}{2}}$;
- τ, transmittance of layer.

Subscripts
- d, diffuse;
- i, internal surface;
- o, external surface;
- s, specular.

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INTRODUCTION

THE RADIATIVE transfer through a scattering, absorbing dielectric sheet has been treated analytically by means of the two-flux model due to the simplicity of the model [1, 2]. This two-flux model assumes the total radiant flux within the layer to be composed of two fluxes with one flux, I, in the direction of positive x-axis and the other flux, J, in the negative x-axis direction. Writing an energy balance on an infinitesimal layer results in a set of two simultaneous differential equations. With appropriate boundary conditions, these two equations can be solved to yield solutions for I and J. Then the transmittance τ, absorptance

α, and reflectivity ρ, of a dielectric sheet of uniform physical and optical properties for diffuse incident radiant energy can be found [2, 3]. For comparison purposes, the existing expressions are written in a slightly different form [2, 3]:

$$\tau = 4\beta_0(1-\rho_0)(1-\rho_i)/F \tag{1}$$

$$\rho = \{2[(1-\rho_i)^2 - \beta^2(1-\rho_i-2\rho_0)(1+\rho_i)]\sinh(\sigma L) + 4\beta(\rho_0+\rho_i)(1-\rho_i)\cdot\cosh(\sigma L)\}/F \tag{2}$$

$$\alpha = 4\beta(1-\rho_0)\{\beta(1+\rho_i)\sinh(\sigma L) + 2(1-\rho_i)[\cosh(\sigma L)-1]\}/F \tag{3}$$

where

$$F \equiv K_1^2\exp(\sigma L) - K_2^2\exp(-\sigma L) \tag{4}$$

$$K_1 \equiv 1+\beta-\rho_i(1-\beta) \tag{5}$$

$$K_2 \equiv 1-\beta-\rho_i(1+\beta) \tag{6}$$

$$\sigma \equiv [a(a+2s)]^{\frac{1}{2}} \tag{7}$$

$$\beta \equiv [a/(a+2s)]^{\frac{1}{2}}. \tag{8}$$

Here a, s and L are, respectively, the absorption coefficient, scattering coefficient and the thickness of the dielectric layer. The external and internal reflectives at the air-dielectric interface are ρ_0 and ρ_i respectively. σ and β defined before are two optical constants, the latter being particularly useful [6] in characterizing a medium since for a strong scattering medium, $\beta \to 0$ and for a lightly scattering medium, $\beta \to 1$.

Numerical values of the external reflectivity as a function of the index of refraction have been tabulated [4]. As to ρ_i, only two asymptotic expressions have been obtained. In the limiting case of strong scattering media (i.e. $\beta \to 0$) it is given by [2, 5]

$$\rho_i = 1 - (1/n^2) + (\rho_0/n^2). \tag{9}$$

For a slightly scattering media (i.e. $\beta \to 1$), Progelhof and Throne [6] have recently shown that the following expression

$$\rho_i = \rho_0 \tag{10}$$

should be used. For example, consider a dielectric sheet with an index of refraction of 1·5 surrounded by air. The internal reflectivity is found to by 0·595 by equation (9) and 0·092 by equation (10). Using these different values to calculate transmittance, reflectance and absorptance given by equations (1)–(3) will certainly yield significantly different results [6]. For the case where both scattering and absorption are important, no appropriate expression of ρ_i is yet available. The current practice is simply to use either equation (9) or (10), which is not satisfactory [6]. Consequently, the conventional method [2, 3] of determining optical constants from equations (1)–(3) and either (9) or (10) by the transmission measurement may not yield accurate results.

It is the purpose of the present study to seek a solution which will be applicable to any possible values of β. The question [6] regarding what value of ρ_i should be used will also be resolved.

ANALYSIS

The physical problem to be considered is similar to that of Klein [2]. There is an incident radiation flux I_i on the front surface of the scattering, absorbing sheet at $x = 0$ and no incident flux on the back surface at $x = L$. According to the two-flux method, the simultaneous differential equations to be solved are

$$dI/dx = -(a+s)I + sJ \tag{11}$$

and

$$dJ/dx = (a+s)J - sI. \tag{12}$$

Since the index of refraction of the dielectric medium is greater than that of the surrounding, all incident radiation transmitted through the surface at $x = 0$ will be confined inside the critical angle $\varphi_c = \sin^{-1}(1/n)$. For a slightly scattering medium, practically all the transmitted radiations will remain inside the critical angle even after they travel through the layer and are reflected back and forth many times. Thus the internal side of the interface behaves like a specular surface with a specular reflectivity $\rho_i = \rho_0$ (see [6]). On the other hand, for a strong scattering medium, the transmitted radiation will be completely diffused as soon as it travels through the layer. Therefore the radiation incident on the internal side of both interfaces will be diffuse and some of the radiation is no longer confined inside the critical angle. Due to the fact that the incident radiation outside the critical angle is totally reflected, the internal reflectivity given by equation (9) should be used.

For the general case of a dielectric medium in which both the scattering and absorption are important, it is reasonable to assume that part of the radiation inside the medium is specular and the other part diffuse. Then, the forward and backward fluxes can be decomposed into two components, i.e.

$$I = I_d + I_s \tag{13}$$

$$J = J_d + J_s \tag{14}$$

where the subscripts d and s indicate diffuse and specular components, respectively. Because of the linearity of the equations involved the differential equations and their appropriate boundary conditions can be decomposed as follows,

$$dI_s/dx = -(a+s)I_s \tag{15}$$

$$dI_d/dx = -(a+s)I_d + s(J_d + J_s) \tag{16}$$

$$dJ_s/dx = (a+s)J_s \tag{17}$$

$$dJ_d/dx = (a+s)J_d - s(I_d + I_s) \tag{18}$$

and

$$\text{at } x = 0,\ I_s = (1-\rho_0)I_i + \rho_s J_s \tag{19}$$

$$I_d = \rho_d J_d \tag{20}$$

$$\text{at } x = L,\ J_s = \rho_s I_s \tag{21}$$

$$J_d = \rho_d I_d \tag{22}$$

where ρ_d and ρ_s are the internal reflectivities given by equations (9) and (10) respectively. It is seen that the specular components, I_s and J_s, are associated with ρ_s and I_d and J_d with p_d. In equation (19), the inclusion of the term $(1-\rho_0)I_i$ in the specular component is because, at $x = 0+$, the transmitted radiation is still confined inside the critical angle. The resulting solution will thus approach the correct existing solution in the limiting case of weakly scattering medium (see the last paragraph).

It should be mentioned that the present approach of separating the fluxes into two parts is somewhat similar to that of Olfe [7] and Landram and Greif [8] yet for different purposes. While their purpose is to account for the isotropic emission from the gas body and the nonisotropic radiation from external sources, the present purpose is to accommodate the different values of internal reflectivity. Such an approach will certainly yield better results than that without separating the fluxes [7, 8].

The general solutions of equations (15)–(18) are

$$I_s = C_1 \exp(-\gamma x), \tag{23}$$

$$J_s = C_2 \exp(\gamma x), \tag{24}$$

$$I_d = A(1-\beta)\exp(\sigma x) + B(1+\beta)\exp(-\sigma x) - C_1 \exp(-\gamma x), \tag{25}$$

and

$$J_d = A(1+\beta)\exp(\sigma x) + B(1-\beta)\exp(-\sigma x) - C_2 \exp(\gamma x) \tag{26}$$

where $\gamma = a + s$. The four constants A, B, C_1 and C_2 can be found by using the boundary conditions equations (19)–(22) as follows:

$$C_1 = \rho_s(1-\rho_0)\exp(-\gamma L)I_i/[\exp(\gamma L) - \rho_s^2 \exp(-\gamma L)] \tag{27}$$

$$C_2 = \rho_s C_1 \exp(-2\gamma L) \tag{28}$$

$$A = [K_{1d}(C_2 \exp(\gamma L) - C_1\rho_d \exp(-\gamma L)) - K_{2d}\exp(-\sigma L)(C_1 - \rho_d C_2)]/F_d \tag{29}$$

$$B = [K_{1d}\exp(\sigma L)(C_1 - \rho_d C_2) - K_{2d}(C_2 \exp(\gamma L) - C_1\rho_d \exp(-\gamma L))]/F_d \tag{30}$$

where F_d, K_{1d} and K_{2d} are identical to the respective expression of F, K_1 and K_2 given in equations (4)–(6) with all the ρ_i's replaced by ρ_d.

Once the forward and backward fluxes are obtained, the transmittance, reflectance and absorptance of the layer can be easily found. The transmittance is defined as the ratio of the amount of radiation that gets through the layer to the incident radiation, or

$$\tau = [(1-\rho_d)I_d|_{x=l} + (1-\rho_s)I_s|_{x=l}]/I_i$$
$$= \frac{(1-\rho_0)(1-\rho_d)\{4\beta[1-\rho_d\rho_s\exp(-2\gamma L)] - (\rho_s-\rho_d)\exp(-\gamma L)\cdot[K_{2d}(1+\beta)\exp(-\sigma L) - K_{1d}(1-\beta)\exp(\sigma L)]\}}{F_d[1-\rho_s^2\exp(-2\gamma L)]} + \frac{(1-\rho_0)(\rho_d-\rho_s)}{\exp(\gamma L) - \rho_s^2\exp(-\gamma L)}. \tag{31}$$

Similarly

$$\rho = [\rho_0 I_i + (1-\rho_d)J_d|_{x=0} + (1-\rho_s)J_s|_{x=0}]/I_i$$
$$= \rho_0 + \frac{(1-\rho_d)(1-\rho_0)\{4\beta(\rho_s-\rho_d)\exp(-\gamma L) - [1-\rho_d\rho_s\exp(-2\gamma L)]\cdot[K_{2d}(1+\beta)\exp(-\sigma L) - K_{1d}(1-\beta)\exp(\sigma L)]\}}{F_d[1-\rho_s^2\exp(-2\gamma L)]} + \frac{\rho_s(1-\rho_0)(\rho_d-\rho_s)\exp(-2\gamma L)}{1-\rho_s^2\exp(-2\gamma L)}, \tag{32}$$

and

$$\alpha = 1 - \tau - \rho$$
$$= \frac{(1-\rho_0)[1-\rho_d\rho_s\exp(-2\gamma L) + (\rho_s-\rho_d)\exp(-\gamma L)]\{F_d + (1-\rho_d)\cdot[K_{2d}(1+\beta)\exp(-\sigma L) - K_{1d}(1-\beta)\exp(\sigma L) - 4\beta]\}}{F_d(1-\rho_s^2\exp(-2\gamma L)]}. \tag{33}$$

RESULTS AND DISCUSSIONS

It can be readily shown that the present expressions of equations (31)–(33) reduce, respectively, to the existing expressions of equations (1)–(3) if the two different internal reflectivities ρ_s and ρ_d are set to be equal to ρ_i. Further check can be made by examining two asymptotic limits. In the case of a lightly scattering medium (i.e. $s \to 0$ or $\beta \to 1$), the asymptotic forms of the present expressions are indeed identical to that of the existing expressions using the appropriate internal reflectance given by equation (10). Similarly, in the case of a strong scattering medium ($s \to \infty$ or $\beta \to 0$), the present asymptotic forms are also identical to the asymptotic forms of the existing expressions using the internal reflectance given by equation (9). Thus the present solutions, which are applicable for any value of β, do agree with these two correct limiting solutions obtained by the previous investigators [2, 5, 6]. The use of the present solution can avoid the intriguing problem regarding which form of the internal reflectance should be used.

It is interesting to show the difference between the present solutions and the existing approximate solutions for intermediate values of β. Figure 1 compares the absorptance

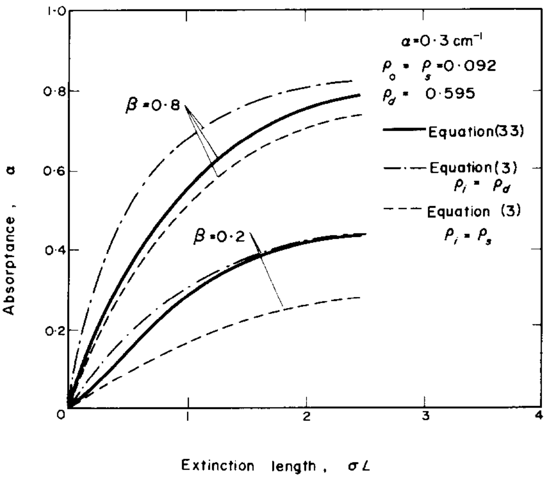

FIG. 1. Absorptance of an absorbing, scattering dielectric layer.

based on the present solution with that based on the existing approximation which simply uses equation (3) with ρ_i evaluated according to equation (9) or (10). It is seen that present results fall between the results of the existing approximation which should provide as the upper and lower bound values. The difference between these upper and lower bound values are significant and the present results can lean to either upper or lower bound value depending on the magnitude of β. In Fig. 2, similar comparison is made for the forward and backward fluxes. The present forward and backward fluxes which are defined respectively by $I = I_d + I_s$ and $J = J_d + J_s$ are compared with the fluxes obtained by Klein [2] using two different internal reflectivities. It should be noted that the forward flux in the layer is not necessarily smaller than the incident flux but the net flux, namely, $I - J$, does.

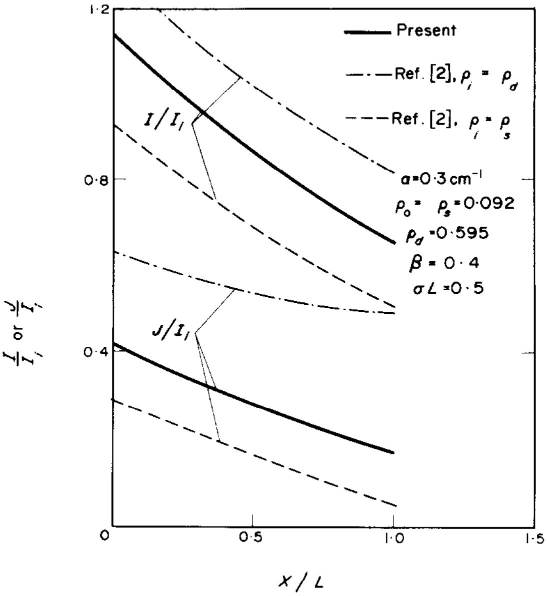

FIG. 2. Forward and backward fluxes in an absorbing, scattering dielectric layer.

The present approach can also be applied to the non-isothermal layer which has been treated by Klein [2]. The procedure is straightforward but final expressions are too lengthy to present here.

CONCLUSION

The radiation characteristics of a scattering, absorbing dielectric layer treated by means of the two-flux model have been re-examined. General solutions were presented for a scattering, absorbing layer with any possible values of β. In the two limiting cases ($\beta \to 0$ and 1) where the correct solutions are available in the literature, the present solutions agree exactly with them. For the intermediate values of β, the present solutions should constitute better approximation than the two asymptotic solutions.

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LAMINAR FORCED CONVECTION HEAT TRANSFER IN CURVED PIPES WITH UNIFORM WALL TEMPERATURE

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NOMENCLATURE

a,	radius of pipe;
C,	constant, $(a^3/4\nu\mu)(\partial P_0/R_c\partial\Omega)$;
$\bar{h}$,	average heat transfer coefficient;
K,	Dean number, $Re(a/R_c)^{\frac{1}{2}}$;
k,	thermal conductivity;
M, N,	number of divisions in R and ϕ directions;
Nu,	Nusselt number, $\bar{h}(2a)/k$;
P_0,	axial pressure measured along the centerline and a function of $R_c\Omega$ only;
Pr,	Prandtl number, ν/α;
Q,	a parameter, $(K^2 Pr)^{\frac{1}{2}}$;
$R, \phi, R_c\Omega$,	cylindrical coordinates;
R_c,	radius of curvature of a curved pipe;
Re,	Reynolds number $(2a)\bar{W}/\nu$;
r,	dimensionless radial coordinate, R/a;
r_c,	dimensionless radius of curvature of a curved pipe, R_c/a;
T,	local temperature;
T_w,	uniform wall temperature;
T_0,	uniform fluid temperature at thermal entrance;
U, V, W,	velocity components in R, ϕ and $R_c\Omega$ directions;
u, v, w,	dimensionless velocity components in r, ϕ and $r_c\Omega$ directions.

Greek letters

α,	thermal diffusivity;
θ,	dimensionless temperature difference;
ν,	kinematic viscosity.

Subscript and superscript

w,	value at wall;
0,	value for straight pipe;
$-$,	average value.

1. INTRODUCTION

THE EXISTING heat transfer results in the literature for fully developed laminar forced convection in curved pipes with uniform wall temperature are rather limited and incomplete in some respects in comparison with the case of uniform wall heat flux [1, 2]. The problem was approached by Maekawa [3] using a perturbation method applicable only to extremely low Dean number flow regime which is practically not important. On the other hand, Mori and Nakayama's approximate solution [4] based on boundary layer approximation near the wall is valid only for high Dean number regime and Prandtl number of order one. Recently, Dravid *et al* [5] presented a numerical result for thermal entrance region heat transfer in curved pipes with uniform wall temperature for the case of Dean number $K = 225$ and $Pr = 5$ only. The purpose of this note is to present an accurate heat transfer result for uniform wall temperature case with Dean number ranging from small to an order of